A SET OF LINEAR EQUATIONS CONNECTED WITH HOMOFOCAL SURFACES.

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1. It is usually stated that the set of equations

$$\left.\begin{array}{l} \dfrac{x_1}{b_1-\beta_1}+\dfrac{x_2}{b_1-\beta_2}+\dfrac{x_3}{b_1-\beta_3}+\dots+\dfrac{x_n}{b_1-\beta_n}=1 \\ \dfrac{x_1}{b_2-\beta_1}+\dfrac{x_2}{b_2-\beta_2}+\dfrac{x_3}{b_2-\beta_3}+\dots+\dfrac{x_n}{b_2-\beta_n}=1 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ \dfrac{x_1}{b_n-\beta_1}+\dfrac{x_2}{b_n-\beta_2}+\dfrac{x_3}{b_n-\beta_3}+\dots+\dfrac{x_n}{b_n-\beta_n}=1 \end{array}\right\}$$

was first solved by Binet* in 1837; and it is certainly true that subsequent to that date additional solutions were given by Chelini, Cauchy, Hädenkamp, and Liouville.† The solution given in Todhunter's "Theory of Equations," where a reference is misleadingly made to Grunert's "Archiv," is really Liouville's.

The object of the present note is to make known a solution which, though simpler than any of those mentioned, is not brought forward wholly on that account, but also in order to draw pointed attention to a paper of Murphy's in which is given a *general mode* of dealing with equations of the type above exemplified, and which bears a date five years prior even to Binet's paper. ‡

* Binet, J., "Observations sur des théorèmes de géométrie . . ." (Journ. (de Liouville) de Math., ii., pp. 248–252).

† Chelini, D., "Formazione e dimostrazione della formula che dà . . ." (Giornale Arcadico lxxxv. (1840), pp. 3–12).

Cauchy, A., "Mémoire sur les fonctions alternées . . ." (Exercices d'analyse et de phys. math., ii. (1841), pp. 151–159).

Hädenkamp, "Ueber Transformation vielfacher Integrale "(Crelle's Journ., xxii. (1841), pp. 184–192).

Liouville, J., "Sur une classe d'équations du premier degré" (Journ. (de Liouville) de Math., xi. (1846), pp. 466–467).

‡ Murphy, R., "On Elimination between an Indefinite Number of Unknown Quantities" (Trans. Cambridge Philos. Soc., v. (1832), pp. 65–76).

2. The essence of the solution consists in noting that to assert the validity of the given set of equations is the same as to say that the expression

$$\frac{x_1}{\xi-\beta_1}+\frac{x_2}{\xi-\beta_2}+\frac{x_3}{\xi-\beta_3}+\ldots+\frac{x_n}{\xi-\beta_n}=1$$

vanishes for n values of ξ, namely, the values $b_1, b_2, \ldots, b_n$; and that therefore

$$\begin{aligned} x_1(\xi-\beta_2)(\xi-\beta_3)\ldots(\xi-\beta_n) \;+\; & x_2(\xi-\beta_1)(\xi-\beta_3)\ldots(\xi-\beta_n) \;+\; \ldots\ldots \\ - \;& (\xi-\beta_1)(\xi-\beta_2)\ldots(\xi-\beta_n) \end{aligned}$$

must be identically equal to

$$A(\xi-b_1)(\xi-b_2)\ldots(\xi-b_n).$$

Since the only term containing ξ^n in the left-hand member is $-\xi^n$, it follows that A is -1. We have only then to put $\xi=\beta_1, \beta_2, \ldots$ in succession in this identity and we obtain

$$x_1=-\frac{(\beta_1-b_1)(\beta_1-b_2)\ldots(\beta_1-b_n)}{(\beta_1-\beta_2)\ldots(\beta_1-\beta_n)}, \quad x_2=\ldots\ldots\ldots$$

3. As a second example, which readily suggests others, let us take the set

$$\left.\begin{aligned} a_1x_1+a_1^2x_2+a_1^3x_3+\ldots+a_1^nx_n&=-1 \\ x_1+2a_1x_2+3a_1^2x_3+\ldots+na_1^{n-1}x_n&=0 \\ a_2x_1+a_2^2x_2+a_2^3x_3+\ldots+a_2^nx_n&=-1 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots& \\ a_{n-1}x_1+a_{n-1}^2x_2+a_{n-1}^3x_3+\ldots+a_{n-1}^nx_n&=-1 \end{aligned}\right\}.$$

Here the equivalent assertion is that the expression

$$1+\xi x_1+\xi^2x_2+\xi^3x_3+\ldots+\xi^nx_n$$

vanishes for $n-1$ values of ξ, namely, the values $a_1, a_2, \ldots, a_{n-1}$, and that its differential-quotient with respect to ξ vanishes for $\xi=a_1$. We thus learn that it is of the form

$$A(\xi-a_1)^2(\xi-a_2)(\xi-a_3)\ldots(\xi-a_{n-1}):$$

and observing that it becomes 1 when ξ is put $=0$ we learn further that

$$A=\frac{(-1)^n}{a_1^2a_2a_3\ldots a_{n-1}}.$$

Our identity thus is

$$1+\xi x_1+\xi^2 x_2+\ldots+\xi^n x_n=\frac{(-1)^n(\xi-a_1)^2(\xi-a_2)(\xi-a_3)\ldots(\xi-a_{n-1})}{a_1^2 a_2 a_3\ldots a_{n-1}},$$

and equating coefficients of like powers of ξ we obtain

$$x_n=\frac{(-1)^n}{a_1^2 a_2 a_3\ldots a_{n-1}},$$

$$x_{n-1}=\frac{(-1)^{n+1}(2a_1+a_2+a_3+\ldots+a_{n-1})}{a_1^2 a_2 a_3\ldots a_{n-1}},$$

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4. The close connection of these with the cases considered by Murphy will be evident when we quote what he calls his "principle," which is, "If we make the right-hand member of the ξth equation disappear by transposition, the left-hand member is then a function of ξ which vanishes when ξ is any member of the series 1, 2, 3, ..., n: and therefore it must be of the form $P(x-1)(x-2)(x-3)\ldots(x-n)$." His cases are those where the ξth equation is

(*a*) $\dfrac{1}{\xi}+\dfrac{x_1}{\xi+1}+\dfrac{x_2}{\xi+2}+\ldots+\dfrac{x_n}{\xi+n}=0,$

(*b*) $1+\xi x_1+\xi(\xi+1)x_2+\ldots+\xi(\xi+1)\ldots(\xi+n-1)x_n=0,$

(*c*) $1+\xi x_1+\xi^2 x_2+\ldots+\xi^n x_n=0.$